NASA Contractor Report 181884
ICASE Report No. 89–58

ICASE

DISCUSSION OF TURBULENCE MODELING: PAST AND FUTURE

Charles G. Speziale

Contract No. NAS1–18605
July 1989

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, Virginia 23665–5225

Operated by the Universities Space Research Association

NASA
National Aeronautics and Space Administration

Langley Research Center
Hampton, Virginia 23665-5225

DISCUSSION OF TURBULENCE MODELING: PAST AND FUTURE

Charles G. Speziale

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, VA 23665

ABSTRACT

The full text of the discussion paper presented at the *Whither Turbulence Workshop* (Cornell University, March 22-24, 1989) on past and future trends in turbulence modeling is provided. It is argued that Reynolds stress models are likely to remain the preferred approach for technological applications for at least the next few decades. In general agreement with the Launder position paper, it is further argued that among the variety of Reynolds stress models in use, second-order closures constitute by far the most promising approach. However, some needed improvements in the specification of the turbulent length scale are emphasized. The central points of the paper are illustrated by examples from homogeneous turbulence.

This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18605 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

Discussion of Turbulence Modeling: Past and Future

Charles G. Speziale

Institute for Computer Applications in Science and Engineering
NASA Langley Research Center
Hampton, VA 23665

1. Introductory Remarks

The thrust of the position paper by Launder is that second-order closure models represent the best hope for the reliable prediction of the complex turbulent flows of technological interest both now and in the foreseeable future. By building on the pioneering research of Rotta [1] and by introducing some fundamental new ideas, the work of Launder, Lumley, and others has without doubt made significant contributions to the advancement of second-order closures. In the position paper by Launder, a strong case is made for the superior predictive capabilities of second-order closures in comparison to two-equation models or eddy viscosity models. Most notably, turbulent flows involving rotations and streamline curvature have been shown by Launder and others [2,3] to be better described by second-order closure models. The same is true for turbulent flows with stratification and relaxation effects. Launder very aptly cites four active areas of research for the improvement of second-order closures: (i) models for the rapid pressure strain correlation, (ii) models for the turbulent diffusion terms, (iii) adjustments for near wall turbulence effects, and (iv) modeled transport equations for the turbulent dissipation rate or length scale.

In the sections to follow, by making use of some simple examples from homogeneous turbulence, the primary point made by Launder concerning the superior predictive capabilities of second-order closures will be amplified. Most notably, it will be shown that second-order closures are capable of describing the stabilizing or destabilizing effect of rotations on shear flow – a problem which cannot be even remotely analyzed by the simpler models. However, some lingering problems concerning the development of adequate models for the rapid pressure-strain correlation and the turbulent length scale will be emphasized (see Speziale [4]). In regard to the latter issue, the strengths and weaknesses of the commonly used modeled dissipation rate transport equation will be discussed and a definitive argument will be put forth as to why previous attempts at the development of an improved dissipation rate transport equation have failed. Alternative approaches based on a tensor length scale will also be discussed along with the author's views concerning the prospects for future research.

2. The Case for Second-Order Closure Models

The commonly used eddy viscosity models and two-equation models have three major deficiencies [5,6]:

(i) the inability to properly account for rotational strains,
(ii) the inaccurate prediction of normal Reynolds stress anisotropies,
(iii) the inability to account for component Reynolds stress relaxation and amplification effects.

In so far as point (i) is concerned, it should be noted that the K- ε model is oblivious to the presence of rotational strains (e.g., it fails to distinguish between the physically distinct cases of plane strain, plane shear, and rotating plane shear). Other commonly used algebraic eddy viscosity models such as the Baldwin-Lomax model are also fundamentally incapable of describing the effect of rotations on sheared or strained turbulent flows [2,6]. As alluded to in point (ii), all eddy viscosity models, including the K-ε model, yield highly inaccurate predictions for the normal Reynolds stress anisotropies in simple turbulent shear flows. This makes it impossible to describe a variety of secondary flow phenomena (e.g., the K-ε model erroneously predicts that there are *unidirectional* mean turbulent flows in non-circular ducts in contradiction to experiments which indicate the presence of an additional secondary flow [5]; see Figure 1). These problems can be partially overcome by the use of two-equation turbulence models with a nonlinear algebraic Reynolds stress model (see Launder and Ying [7], Rodi [8], and Speziale [5]), but only for turbulent flows that are nearly in equilibrium. Non-equilibrium turbulent flows that have a spatially or temporally evolving structure (e.g., the flows with relaxation or amplification effects mentioned in point (iii)) cannot, in general, be described properly by two-equation models. For example, in an initially anisotropic turbulence, where at some time $t = t_0$ the mean velocity gradients are set to zero, the K-ε model erroneously predicts an instantaneous return to isotropy wherein

$$\tau_{ij} = -\frac{2}{3}K\delta_{ij}, \quad t \geq t_0 \tag{1}$$

or equivalently,

$$b_{ij} = 0, \quad t \geq t_0 \tag{2}$$

given that K is the turbulent kinetic energy, $\tau_{ij} \equiv -\overline{u_i u_j}$ is the Reynolds stress tensor, and $b_{ij} \equiv -(\tau_{ij} + \frac{2}{3}K\delta_{ij})/2K$ is the anisotropy tensor. In considerable contradiction to (1), experiments indicate that there is a very gradual return to isotropy – an effect that can be characterized much better by second-order closure models. In Figure 2, the temporal evolution of the second invariant of the anisotropy tensor II is shown corresponding to a relaxation from the plane strain experiment of Choi and Lumley [9]. From this graph, it is clear that the second-order closure model (which is based on the Rotta model for the slow pressure-strain correlation) does a reasonably good job in reproducing the experimental trends unlike the K-ε model which erroneously predicts that $II = 0$ for dimensionless time $\tau \geq 0$.

Now, the greater predictive capabilities of second-order closure models will be demonstrated by a simple, but non-trivial, example which is not often discussed in the turbulence modeling literature. The problem to be considered is homogeneous turbulent shear flow in a rotating frame (see Figure 3). This problem constitutes a non-trivial test of turbulence models since it involves arbitrary combinations of shear and rotation which can have either a stabilizing or destabilizing effect.

For any homogeneous turbulent flow, the standard K-ε model takes the general form [10]

$$\tau_{ij} = -\frac{2}{3}K\delta_{ij} + C_\mu \frac{K^2}{\varepsilon}\left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i}\right) \tag{3}$$

$$\dot{K} = \tau_{ij}\frac{\partial \overline{v}_i}{\partial x_j} - \varepsilon \tag{4}$$

$$\dot{\varepsilon} = C_{\varepsilon 1}\frac{\varepsilon}{K}\tau_{ij}\frac{\partial \overline{v}_i}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon^2}{K} \tag{5}$$

in all frames of references independent of whether or not they are inertial. In (3)-(5), $\overline{v}_i$ is the mean velocity field, ε is the turbulent dissipation rate, and $C_\mu, C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are constants which assume the values of 0.09, 1.44, and 1.92, respectively. For homogeneous turbulent shear flow in a rotating frame (as shown in Figure 3), the mean velocity gradient tensor is given by

$$\frac{\partial \overline{v}_i}{\partial x_j} = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

and $\Omega_i = (0, 0, \Omega)$ is the rotation rate of the reference frame relative to an inertial framing. Since (3)-(5) are independent of Ω, the standard K-ε model predicts the *same results for all rotation rates* and, hence, does not distinguish between turbulent shear flow in an inertial frame and rotating turbulent shear flow. Speziale and Mac Giolla Mhuiris [11] recently showed that the K-ε model has the following equilibrium solution for rotating shear flow:

$$(b_{11})_\infty = 0, \quad (b_{12})_\infty = -\frac{1}{2}(C_\mu \alpha)^{\frac{1}{2}}, \quad (b_{13})_\infty = 0 \tag{7}$$

$$(b_{22})_\infty = 0, \quad (b_{23})_\infty = 0, \quad (b_{33})_\infty = 0 \tag{8}$$

$$\left(\frac{SK}{\varepsilon}\right)_\infty = \left(\frac{\alpha}{C_\mu}\right)^{\frac{1}{2}} \tag{9}$$

where $\alpha = (C_{\varepsilon 2} - 1)/(C_{\varepsilon 1} - 1)$ and $(\cdot)_\infty$ denotes the equilibrium value obtained in the limit as $t \to \infty$. These equilibrium values are universal, i.e., are completely independent of the initial conditions, the shear rate, and the rotation rate. It was also shown in [11] that the long time solutions (corresponding to $t^* \equiv St \gg 1$) for the kinetic energy and dissipation rate in the K-ε model grow exponentially:

$$K \sim \exp\left[\sqrt{\frac{C_\mu}{\alpha}}(\alpha - 1)t^*\right] \tag{10}$$

$$\varepsilon \sim \exp\left[\sqrt{\frac{C_\mu}{\alpha}}(\alpha - 1)t^*\right] \tag{11}$$

Hence, the K-ε model predicts the following physical picture for rotating shear flow: the turbulent kinetic energy and dissipation rate grow exponentially in time at a comparable rate; the anisotropy tensor b_{ij} and shear parameter SK/ε approach a universal equilibrium. While this characterization is qualitatively correct for pure shear flow (see Tavoularis and Corrsin [12]), it is quite incorrect for most values of Ω/S in rotating shear flow. Linear stability analyses and numerical simulations of the Navier-Stokes equations indicate that for values of the rotation rate that are discernibly outside of the range $0 \le \Omega/S \le 0.5$, the flow undergoes a restabilization wherein K and $\varepsilon \to 0$ as $t \to \infty$ (see Bardina, Ferziger, and Reynolds [13] and Bertoglio [14]).

It will now be demonstrated that, unlike the commonly used two-equation models, second-order closures are able to describe the stabilizing or destabilizing effect of rotations on turbulent shear flow. Speziale and Mac Giolla Mhuiris [11] recently considered a fairly general class of second-order closure models of the form:

$$\dot{\tau}_{ij} = -\tau_{ik}\frac{\partial \overline{v}_j}{\partial x_k} - \tau_{jk}\frac{\partial \overline{v}_i}{\partial x_k} - \Pi_{ij} + \frac{2}{3}\varepsilon\delta_{ij} - 2(\tau_{ik}e_{mkj}\Omega_m + \tau_{jk}e_{mki}\Omega_m) \tag{12}$$

$$\dot{\varepsilon} = C_{\varepsilon 1}\frac{\varepsilon}{K}\tau_{ij}\frac{\partial \overline{v}_i}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon^2}{K} \tag{13}$$

for any rotating homogeneous turbulence. In (12)-(13), Π_{ij} denotes the pressure-strain correlation which is assumed to be of the general form

$$\Pi_{ij} = \Pi_{ij}(\tau_{ij}, \frac{\partial \overline{v}_i}{\partial x_j}, \varepsilon) \tag{14}$$

and $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ are either constants or functions of the invariants of b_{ij}. This class of second-order closures encompasses a wide variety of models including the simplified form of the Launder, Reece, and Rodi model for which

$$\begin{aligned}\Pi_{ij} \;=\; & C_1 \frac{\varepsilon}{K}(\tau_{ij} + \frac{2}{3}K\delta_{ij}) - C_2 \left[\tau_{ik}\left(\frac{\partial \overline{v}_j}{\partial x_k} + \varepsilon_{mkj}\Omega_m\right)\right. \\ & \left. + \tau_{jk}\left(\frac{\partial \overline{v}_i}{\partial x_k} + \varepsilon_{mki}\Omega_m\right)\right] + \frac{2}{3}C_2 \tau_{mn}\frac{\partial \overline{v}_m}{\partial x_n}\delta_{ij}\end{aligned} \tag{15}$$

and $C_1 = 1.8$, $C_2 = 0.6$, $C_{\varepsilon 1} = 1.44$ and $C_{\varepsilon 2} = 1.92$. It was shown by Speziale and Mac Giolla Mhuiris [11] that this class of second-order closure models has two-equilibrium solutions for rotating shear flow: one where

$$\left(\frac{\varepsilon}{SK}\right)_\infty = 0 \tag{16}$$

which exists for *all* Ω/S and one where

$$\left(\frac{\varepsilon}{SK}\right)_\infty = \gamma_0 \left[\gamma_1 + \gamma_2\left(\frac{\Omega}{S}\right) - \left(\frac{\Omega}{S}\right)^2\right]^{\frac{1}{2}} \tag{17}$$

which exists for a small intermediate band of Ω/S which can range from $-0.1 \leq \Omega/S \leq 0.6$ (here, γ_0, γ_1, and γ_2 are directly related to the constants of the model). The former equilibrium solution for which $(\varepsilon/SK)_\infty = 0$, predominantly is connected with solutions wherein K and ε undergo a power law decay in time; the latter equilibrium solution (17), where $(\varepsilon/SK)_\infty$ is nonzero, is connected with solutions where K and ε grow exponentially in time at the same rate. In this intermediate band of Ω/S, these two solutions exchange stabilities in a fashion that qualitatively mimics the shear instability with its exponential time growth of disturbance kinetic energy.

In Figure 4(a), a bifurcation diagram is shown for the Launder, Reece, and Rodi model. This bifurcation structure qualitatively mimics the stabilizing or destabilizing effects of rotations on turbulent shear flow as discussed above. In stark contrast to the bifurcation that is properly predicted by the second-order closure, the equilibrium diagram for the K-ε model shows the erroneous prediction of a universal value for $(\varepsilon/SK)_\infty$ which is completely independent of Ω/S (see Figure 4(b)). As mentioned earlier, this universal equilibrium solution for the K-ε model corresponds to an unstable flow wherein K and ε grow exponentially in time.

In addition to yielding a superior qualitative description of the equilibrium structure of rotating turbulent shear flows, the quantitative values of the equilibrium states for pure shear flow predicted by the second-order closures are also substantially better than those obtained from the commonly used two-equation models. To illustrate this superiority of the second-order closures, the equilibrium values for b_{ij} and SK/ε obtained from the Launder, Reece, and Rodi model and the K-ε model are compared in Table 1 with the experimental results of Tavoularis and Corrsin [12] for homogeneous turbulent shear flow.

In Figures 5(a)-(c), the time evolution of the turbulent kinetic energy predicted by the Launder, Reece, and Rodi model and the K-ε model are compared with results from the large-eddy simulations of Bardina, Ferziger, and Reynolds [13] for three rotation rates: $\Omega/S = 0$, $\Omega/S = 0.25$, and $\Omega/S = -0.5$. A direct comparison of Figure 5(a) and Figure 5(b) with the large-eddy simulations shown in Figure 5(c), graphically demonstrates the superior capability of second-order closure models in predicting the stabilizing or destabilizing effect of rotations on shear flow.

3. Needed Modeling Improvements in Second-Order Closures

While the author is in full agreement with the main points of the Launder position paper concerning the superior capabilities of second-order closure models, it must be cautioned that these models have not yet matured to the point where reliable predictions can be made for a *variety* of complex turbulent flows. Several areas where improvements are needed (some of which were pointed out by Professor Launder), will be discussed in more detail in this section.

Second-order closure models are based on the Reynolds stress transport equation which takes the exact form

$$\frac{\partial \tau_{ij}}{\partial t} + \bar{v}_k \frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik}\frac{\partial \bar{v}_j}{\partial x_k} - \tau_{jk}\frac{\partial \bar{v}_i}{\partial x_k} + \frac{\partial C_{ijk}}{\partial x_k} - \Pi_{ij} + \varepsilon_{ij} + \nu\nabla^2\tau_{ij} \tag{18}$$

where

$$C_{ijk} \equiv \overline{u_i u_j u_k} + \overline{pu_i}\delta_{jk} + \overline{pu_j}\delta_{ik} \tag{19}$$

$$\Pi_{ij} \equiv \overline{p(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})} \tag{20}$$

$$\varepsilon_{ij} \equiv 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \tag{21}$$

are the third-order diffusion correlation, the pressure-strain correlation, and the dissipation rate correlation, respectively (ν is the kinematic viscosity of the fluid). In order for closure to be achieved at this "second moment" level (which forms the raison d'etre of second-order modeling), models must be developed wherein the higher-order correlations C_{ijk}, Π_{ij} and ε_{ij} are taken to be functionals of the Reynolds stress τ_{ij}, mean velocity gradients $\partial \bar{v}_i/\partial x_j$ and some length scale of turbulence Λ. The Reynolds stress is decomposed into isotropic and deviatoric parts as follows

$$\tau_{ij} = -\frac{2}{3}K\delta_{ij} - 2Kb_{ij} \tag{22}$$

and the turbulence length scale Λ is usually assumed to be of the form

$$\Lambda = C^* \frac{K^{\frac{3}{2}}}{\varepsilon} \tag{23}$$

where C^* is a dimensionless constant and $\varepsilon \equiv \frac{1}{2}\varepsilon_{ii}$ is the turbulent dissipation rate. Hence, consistent with the use of (23), the higher-order correlations can be taken to be functionals of $b_{ij}, \partial \bar{v}_i/\partial x_j, K$, and ε instead.

Typically, the third-order diffusion correlation is modeled by a gradient transport hypothesis wherein it is assumed that C_{ijk} is of the general form

$$C_{ijk} = C_{ijklmn}(\mathbf{b}, K, \varepsilon)\frac{\partial \tau_{\ell m}}{\partial x_n}. \tag{24}$$

Motivated by analyses based on homogeneous turbulence [15], virtually all of the commonly used models for the pressure-strain correlation are assumed to be of the form

$$\overline{p(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})} = \varepsilon A_{ij}(\mathbf{b}) + KM_{ijk\ell}(\mathbf{b})\frac{\partial \bar{v}_k}{\partial x_\ell} \tag{25}$$

where the first term on the right-hand-side of (25) represents the slow pressure strain while the second term represents the rapid pressure strain. Typically, the turbulence dissipation correlation ε_{ij} is assumed to be of the general form

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij} + f_s\varepsilon b_{ij} \tag{26}$$

where f_s is taken to be dimensionless constant or a function of the invariants of b_{ij}. For high Reynolds number turbulent flows that are sufficiently far from solid boundaries, f_s is taken to be zero. In order to achieve closure of the Reynolds stress transport equation, (24)-(26) must be supplemented with a modeled transport equation for the turbulent dissipation rate which is of the general form

$$\frac{\partial \varepsilon}{\partial t} + \bar{v}_k\frac{\partial \varepsilon}{\partial x_k} = \nu\nabla^2\varepsilon + \mathcal{P}_\varepsilon - \Phi_\varepsilon + \mathcal{D}_\varepsilon \tag{27}$$

where $\mathcal{P}_\varepsilon$, Φ_ε and $\mathcal{D}_\varepsilon$ represent the production, dissipation, and turbulent diffusion of ε.

Most of the existing second-order closure models can be constructed by expanding the unknown tensor coefficients on the right-hand-sides of (24)-(25) in a Taylor series in

b_{ij} (subject to the symmetry properties of C_{ijk} and Π_{ij}). The older models are actually first-order Taylor expansions in b_{ij}; for example in the Launder, Reece, and Rodi model [16]:

$$C_{ijk} = -C_s \frac{K}{\varepsilon}(\tau_{i\ell}\frac{\partial \tau_{jk}}{\partial x_\ell} + \tau_{j\ell}\frac{\partial \tau_{ik}}{\partial x_\ell} + \tau_{k\ell}\frac{\partial \tau_{ij}}{\partial x_\ell}) \tag{28}$$

$$\begin{aligned} \Pi_{ij} \quad = \quad & -C_1 \varepsilon b_{ij} + C_2 K \overline{S}_{ij} + C_3 K (b_{ik}\overline{S}_{jk} \\ & + b_{jk}\overline{S}_{ik} - \tfrac{2}{3} b_{mn}\overline{S}_{mn}\delta_{ij}) + C_4 K (b_{ik}\overline{W}_{jk} + b_{jk}\overline{W}_{ik}) \end{aligned} \tag{29}$$

where $\overline{S}_{ij} \equiv \frac{1}{2}(\partial \overline{v}_i/\partial x_j + \partial \overline{v}_j/\partial x_i)$ and $\overline{W}_{ij} = \frac{1}{2}(\partial \overline{v}_i/\partial x_j - \partial \overline{v}_j/\partial x_i)$ are the mean rate of strain tensor and vorticity tensor, respectively. In the simplified version of the Launder, Reece, and Rodi model (which is now referred to as the "Basic Model" by Launder and his co-workers), $C_1 = 1.8$, $C_2 = 0.8$, $C_3 = 1.2$, $C_4 = 1.2$, and $C_s = 0.11$.

The modeled terms in the dissipation rate transport equation (27) are typically based on the assumption that the production (or dissipation) of the turbulent dissipation is proportional to the production (or dissipation) of the turbulent kinetic energy. A gradient transport hypothesis is typically invoked for the turbulent diffusion term on the right-hand-side of (27). These assumptions give rise to a modeled transport equation for the turbulent dissipation rate that takes the general form

$$\begin{aligned} \frac{\partial \varepsilon}{\partial t} + \overline{v}_i \frac{\partial \varepsilon}{\partial x_i} \quad = \quad & C_{\varepsilon 1}\frac{\varepsilon}{K}\tau_{ij}\frac{\partial \overline{v}_i}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon^2}{K} \\ & -\frac{\partial}{\partial x_i}(C_\varepsilon \frac{K}{\varepsilon}\tau_{ij}\frac{\partial \varepsilon}{\partial x_j}) + \nu \nabla^2 \varepsilon. \end{aligned} \tag{30}$$

In the Launder, Reece, and Rodi model, $C_{\varepsilon 1}$, $C_{\varepsilon 2}$, and C_ε are taken to be constants which assume the values of 1.44, 1.92, and 0.15, respectively. Some more recent models have taken $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ to be functions of some subset of the invariants of b_{ij} and $\partial \overline{v}_i/\partial x_j$; these newer models will be discussed in more depth in the next section.

Now, with the aid of this background material, the four active areas of research for the development of improved second-order closure models that were mentioned in the introductory remarks can be elaborated on. These areas are as follows:

(i) The development of improved models for the pressure-strain correlation of turbulence which account for nonlinear anisotropic effects. Since for most flows of engineering interest, $||\mathbf{b}|| \sim 0.2$, the use of a first-order Taylor expansion in $\mathbf{b}$ for A_{ij} and $M_{ijk\ell}$ is highly questionable. In fact, nonlinear terms in the model for A_{ij} are needed to predict the curved trajectories that occur in the phase space of the return to isotropy problem as shown in Figure 6 (see Choi and Lumley [9] and Sarkar and Speziale [17]). At least a quadratic nonlinearity in the model for $M_{ijk\ell}$ is needed in order to satisfy the constraint of Material Frame Indifference (MFI) in the limit of two-dimensional turbulence (Speziale [18] and Haworth and Pope [19]). This MFI constraint is the mathematical embodiment of the well-known result that two-dimensional disturbances evolve *identically* in both a rotating frame and an inertial frame.* In addition, recent work on rotating turbulent shear flows (Speziale, Sarkar, and Gatski [20]) and Rapid Distortion Theory (Reynolds [21]) have suggested the possible need for new terms in (25) that are nonlinear in the mean velocity gradients $\partial \bar{v}_i / \partial x_j$.

(ii) The development of non-gradient transport models for the turbulent diffusion terms such as C_{ijk} need to be considered seriously. It is well-known that turbulent flows do not have a clear cut separation of scales; the largest eddies (which contain a significant portion of the turbulent kinetic energy) are of a comparable size to the geometrical scale of the flow. Consequently, one would expect a gradient transport hypothesis (which is rigorously derived as a first-order expansion in the ratio of fluctuating to mean length scales) to only constitute a crude approximation. Many difficulties in the prediction of turbulent mixing layers could be tied to this deficiency in the modeling of turbulent diffusion by means of gradient transport.

(iii) The development of asymptotically consistent near wall corrections to the turbulence models for Π_{ij}, ε_{ij} and C_{ijk} that are geometry-independent and do not have any ad hoc damping functions. Most of the commonly used corrections are either asymptotically inconsistent, geometry-dependent through an artificially imposed dependence on the unit normal to the wall, or otherwise ad hoc through the use of wall damping functions based on the turbulence Reynolds number or the distance from the wall (see Hanjalic and Launder [22]). Such empiricisms do not allow for the reliable prediction of wall transport properties (e.g., skin friction and heat transfer coefficients) that are extremely important in aerodynamic applications.

(iv) The development of improved modeled transport equations for the turbulence length scale is an issue of utmost importance. Most of the commonly used models have a scalar turbulence length scale Λ based on the dissipation ($\Lambda \propto K^{\frac{3}{2}}/\varepsilon$). There is the obvious objection that the construction of a turbulence macro-scale based on small-scale (one-point) information is conceptually wrong. Furthermore, this rather simplified definition of length scale contains *no directional information.* Although attempts have been made to develop a length scale equation based on an integral of the two-point velocity correlation tensor (Wolfshtein [23] and Donaldson and Sandri [24]), it can be shown that

*Consequently, the models that violate this constraint cannot be used in the analysis of geostrophic turbulence.

these specific models are equivalent to the standardly used dissipation rate transport model in the limit of homogeneous turbulence – a simplified case for which the modeled dissipation rate transport equation is already deficient. Fundamentally new research is needed on the development of modeled transport equations for some appropriate choice of integral length scales that contain the required directional information.

Finally, in regard to these four points, it should be mentioned that some significant progress has been made in the development of improved models for the rapid pressure-strain correlation by means of realizability (Lumley [25], and Shih and Lumley [26]), and the invariance considerations discussed in point (i). However, in the opinion of the author, very little progress, if any, has been made since the early 1970's in the development of better models for the turbulence length scale and diffusion effects. An examination of the modeled dissipation rate transport equation as a basis for the turbulence length scale will be discussed in the next section.

4. The Modeled Dissipation Rate Transport Equation

Now, the strengths and weaknesses of the commonly used modeled dissipation rate transport equation will be discussed. Furthermore, an attempt will be made to demonstrate at what level of approximation this model is derivable from the two-point correlation tensor which more properly contains information about the turbulent macroscale. Finally, an argument will be put forth as to why attempts at the development of improved modeled dissipation rate transport equations have failed during the past decade.

In order not to cloud the issue with the added difficulties that are associated with the integration of turbulence models to a solid boundary, the more simplified case of homogeneous turbulence will be considered. For any homogeneous turbulence, the standardly used version of the modeled dissipation rate transport equation (30) reduces to

$$\dot{\varepsilon} = C_{\varepsilon 1}\frac{\varepsilon}{K}\tau_{ij}\frac{\partial \overline{v}_i}{\partial x_i} - C_{\varepsilon 2}\frac{\varepsilon^2}{K}. \tag{31}$$

The transport equation for the turbulent kinetic energy has the exact form

$$\dot{K} = \tau_{ij}\frac{\partial \overline{v}_i}{\partial x_j} - \varepsilon \tag{32}$$

and is obtained from a contraction of (18). Equations (31)-(32) can be combined to yield a transport equation for the reciprocal turbulent time scale ε/K which is given by

$$\frac{d}{dt}\left(\frac{\varepsilon}{K}\right) = 2(1 - C_{\varepsilon 1})\frac{\varepsilon}{K}b_{ij}\frac{\partial \overline{v}_i}{\partial x_j} + (1 - C_{\varepsilon 2})\left(\frac{\varepsilon}{K}\right)^2. \tag{33}$$

Provided that $C_{\varepsilon 1} > 1$, this equation has two equilibrium solutions in the limit as $t \to \infty$:

$$\left(\frac{\varepsilon}{K}\right)_\infty = 0 \tag{34}$$

$$\left(\frac{\varepsilon}{K}\right)_\infty = -2\left(\frac{C_{\varepsilon 1}-1}{C_{\varepsilon 2}-1}\right)(b_{ij})_\infty \frac{\partial \overline{v}_i}{\partial x_j} \tag{35}$$

which are obtained by setting the time derivative on the left-hand-side of (33) to zero. $C_{\varepsilon 1} = 1$ constitutes a bifurcation point of equation (33); for $C_{\varepsilon 1} \leq 1$, the only realizable fixed point of (33) is $(\varepsilon/K)_\infty = 0$. The zero fixed point (34) predominantly corresponds to solutions for K and ε that undergo a power law decay in time [11] (i.e., ε and $K \to 0$ as $t \to \infty$). This branch of solutions allows for the prediction of isotropic decay as well as the flow restabilization that occurs at certain rotation rates in rotating plane shear and plane strain turbulence. Furthermore, since from (33) we have

$$\frac{d}{dt}\left(\frac{\varepsilon}{K}\right) = 0 \tag{36}$$

when $\varepsilon/K = 0$, and from (31) we have

$$\frac{d\varepsilon}{dt} = 0 \tag{37}$$

when $\varepsilon = 0$, we conclude that if $\varepsilon(0) > 0$ and $K(0) > 0$ then $\varepsilon(t) \geq 0$ and $K(t) \geq 0$ for all later times t. *Hence, the standardly used modeled ε-transport equation guarantees realizability with respect to K and ε by virtue of the fact that $\varepsilon/K = 0$ is, in dynamical systems terms, an "invariant plane."*

The non-zero fixed point (35) is associated with solutions for K and ε which grow exponentially in time [11]. More precisely, for the non-zero fixed point (35), it can be shown that

$$K \sim \exp(\lambda t), \;\; \varepsilon \sim \exp(\lambda t) \tag{38}$$

for $\lambda t \gg 1$, where

$$\lambda = |2(b_{ij})_\infty \frac{\partial \overline{v}_i}{\partial x_j} + \left(\frac{\varepsilon}{K}\right)_\infty|. \tag{39}$$

This allows for the prediction of a structural equilibrium in homogeneous turbulent shear flow wherein SK/ε achieves an equilibrium value that is independent of both the initial conditions and the shear rate. Such an equilibrium for turbulent shear flows has been observed experimentally by Tavoularis and Corrsin [12] for weak to moderately strong shear rates.

It is thus clear that the two fixed points (34)-(35) of the commonly used ε-transport equation have certain properties that are crucial to the proper description of homogeneous turbulent flows. Now, it will be shown that recently proposed alterations to this modeled ε-transport equation destroyed one or the other of these key fixed points and, hence, were doomed to failure.

For example, based on the desire to account for rotational strains, Pope [27] proposed an alteration to the ε-transport equation wherein a term of the form

$$C_{\varepsilon 3}\frac{K^2}{\varepsilon}\overline{S}_{ij}\overline{W}_{jk}\overline{W}_{ki}$$

(where $C_{\varepsilon 3}$ is a constant) was added to the right-hand-side of (30). This eliminates the $(\varepsilon/K)_\infty = 0$ fixed point, for nonzero $\overline{S}_{ij}\overline{W}_{jk}\overline{W}_{ki}$, which can cause problems with realizability and can eliminate the ability to predict flow restabilizations in three-dimensional rotating shear flows. Hanjalic and Launder [28] proposed a modification wherein the term

$$-C_{\varepsilon 3}K\overline{W}_{ij}\overline{W}_{ij}$$

was added to the right-hand-side of (30) (where $C_{\varepsilon 3}$ is a different constant). It is a simple matter to show that, for any nonzero $\overline{W}_{ij}$, this alteration also eliminates the fixed point $(\varepsilon/K)_\infty = 0$. Consequently, this model can have problems with realizability and predicts that the turbulent kinetic energy and dissipation rate oscillates (for all times $t > 0$) in rotating isotropic turbulence – a prediction that is in substantial contradiction to the results of physical and numerical experiments which suggest a monatonic power law decay. Most recently, Launder and his co-workers proposed a modified version of the ε-transport equation for which

$$C_{\varepsilon 1} = 1,\ C_{\varepsilon 2} = C_{\varepsilon 2}(II, III). \tag{40}$$

In this case, the choice of $C_{\varepsilon 1} = 1$ eliminates the nonzero fixed point (35) (the only fixed point is $(\varepsilon/K)_\infty = 0$). Consequently, this model will not predict an exponential time growth of K and ε in homogeneous shear flow (and SK/ε will not approach a universal equilibrium) in apparent contradiction to physical and numerical experiments. Similar problems occur in rotating turbulent shear flows with a modification proposed recently by Bardina, Ferziger, and Rogallo [29].

It has been demonstrated that the standard form of the modeled dissipation rate transport equation has several crucial properties that have been destroyed by virtually every attempt to modify it. To understand this point more deeply, it would be helpful to see what approximations are necessary to derive the modeled ε-transport equation (31) from an analysis of the dynamics of the two-point correlation tensor

$$R_{ij} \equiv \overline{u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})}. \tag{41}$$

A study conducted not too long ago by Donaldson and Sandri [24] can shed some light on this issue. They introduced a tensor length scale Λ_{ij} defined by the solid angle integration

$$\frac{2}{3}K\Lambda_{ij} = \int_{-\infty}^{\infty} \frac{R_{ij}}{4\pi r^2} d^3 r. \tag{42}$$

A modeled transport equation for Λ_{ij} was obtained as follows: (a) the exact transport equation for R_{ij} was closed by assuming that the higher-order unknown correlations could be expanded in a Taylor series in R_{ij} (subject to the appropriate symmetry properties) where only first-order terms were kept, and (b) the resulting modeled transport equation for R_{ij} was multiplied by $1/4\pi r^2$ and integrated over all of $\mathbf{r}$ space. The derived transport equation for Λ_{ij} obtained by this method is as follows for homogeneous turbulence [24]:

$$\begin{aligned} \frac{d}{dt}(K\Lambda_{ij}) &= -K\Lambda_{ik}\frac{\partial \overline{v}_j}{\partial x_k} - K\Lambda_{jk}\frac{\partial \overline{v}_i}{\partial x_k} \\ &\quad +\sqrt{2}v_{c_2}\frac{K^{\frac{3}{2}}}{\Lambda}\Lambda_{ij} - \sqrt{2}\frac{K^{\frac{3}{2}}}{\Lambda}(\Lambda_{ij} - \Lambda\delta_{ij}) \\ &\quad -2\sqrt{2}\, bK^{\frac{3}{2}}\delta_{ij} \end{aligned} \tag{43}$$

where $\Lambda \equiv \frac{1}{3}\Lambda_{ii}$; v_{c_2} and b are constants. The modeled Reynolds stress transport equation that is solved in conjunction with (43) is obtained from a simple contraction of the modeled transport equation for R_{ij}; this equation takes the form [24]

$$\begin{aligned} \frac{d\tau_{ij}}{dt} &= -\tau_{ik}\frac{\partial \overline{v}_j}{\partial x_k} - \tau_{jk}\frac{\partial \overline{v}_i}{\partial x_k} - \sqrt{2}\frac{K}{\Lambda}(\tau_{ij} + \frac{2}{3}K\delta_{ij}) \\ &\quad +\frac{4}{3}\sqrt{2}\, b\frac{K^{\frac{3}{2}}}{\Lambda}\delta_{ij}. \end{aligned} \tag{44}$$

From a direct comparison of the contraction of (44) with (32), it follows that

$$\varepsilon = 2\sqrt{2}\, b\frac{K^{\frac{3}{2}}}{\Lambda} \tag{45}$$

for this tensor length scale model in homogeneous turbulence. By taking the time derivative of (45), the transport equation

$$\begin{aligned} \dot{\varepsilon} &= 2\sqrt{2}\, b(\frac{3}{2}\frac{K^{\frac{1}{2}}}{\Lambda}\dot{K} - \frac{K^{\frac{3}{2}}}{\Lambda^2}\dot{\Lambda}) \\ &= \frac{1}{2}\frac{\varepsilon}{K}\tau_{ij}\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\sqrt{2}}{6b}\frac{\varepsilon^2}{K^{\frac{3}{2}}}\Lambda_{ij}\frac{\partial \overline{v}_i}{\partial x_j} \\ &\quad -(\frac{3}{2} + \frac{1}{2}\frac{v_{c2}}{b})\frac{\varepsilon^2}{K} \end{aligned} \tag{46}$$

is obtained. If an anisotropy tensor is defined for the tensor length scale in the following manner

$$b_{ij}^{(\Lambda)} \equiv \frac{\Lambda_{ij} - \frac{1}{3}\Lambda_{kk}\delta_{ij}}{\Lambda_{kk}} \tag{47}$$

it can be shown that $b_{ij}^{(\Lambda)}$ is a solution of a transport equation that is of *exactly the same form* as that for b_{ij}. Hence, if we consider a homogeneous turbulence that evolves from an initially isotropic state, then

$$b_{ij}^{(\Lambda)} = b_{ij} \tag{48}$$

and

$$\Lambda_{ij}\frac{\partial \overline{v}_i}{\partial x_j} = 3\sqrt{2}\, b\frac{K^{\frac{1}{2}}}{\varepsilon}\tau_{ij}\frac{\partial \overline{v}_i}{\partial x_j}. \tag{49}$$

Then the dissipation rate transport equation (46) corresponding to this tensor length scale model simplifies to

$$\dot{\varepsilon} = 1.5\frac{\varepsilon}{K}\tau_{ij}\frac{\partial \overline{v}_i}{\partial x_j} - \frac{3}{2}(1 + \frac{v_{c_2}}{3b})\frac{\varepsilon^2}{K} \tag{50}$$

which is of the same form as the standardly used equation (31) with $C_{\varepsilon 1} = 1.5$ and $C_{\varepsilon 2} = (\frac{3}{2} + v_{c_2}/2b) \approx 1.80$.

It has thus been shown that the standard form of the modeled ε-transport equation can be derived from the two-point correlation tensor by making approximations that are comparable to those made in the derivation of the modeled Reynolds stress transport equation by Launder, Reece, and Rodi [16]. Consequently, it seems to be somewhat questionable to argue (as many have done) that the standard form of the ε-transport equation is the weak link in the commonly used second-order closures. There is no doubt that improvements in the specification of the turbulence length scale are needed, but these are more likely to come from general analyses of some set of modeled transport equations for the integral length scales which incorporate directional information. The kind of ad hoc adjustments in the modeled dissipation rate transport equation that have been considered during the past decade appear to be counterproductive.

5. Concluding Remarks

Projected advances in computer capacity make it highly unlikely, in the opinion of the author, that the complex turbulent flows of engineering interest will be solved on a routine basis by direct or large-eddy simulations for at least several decades to come. Furthermore, substantial theoretical difficulties with two-point closures for complex turbulent flows that are strongly inhomogeneous make their application to problems of engineering interest equally unlikely for the foreseeable future. It appears that Reynolds stress models are likely to remain the method of choice for the solution of the turbulence problems of technological interest for at least the next few decades. Direct numerical simulations of the Navier-Stokes equations will continue to be restricted to less geometrically complex turbulent flows, at lower Reynolds numbers, where they will be used to gain a better insight into the basic physics of turbulence.

Among the variety of existing Reynolds stress closures, it is the opinion of the author that second-order closure models constitute, by far, the most promising approach. The predictive capabilities of second-order closures have been enhanced somewhat over the past decade due to significant modeling improvements in the pressure-strain correlation. However, new prescriptions for the turbulent length scale (which incorporate some directional and two-point information) as well as new methods for integration to a solid boundary are direly needed before more reliable models can be obtained. In fact, these issues are of such overriding importance for wall-bounded turbulent flows that Reynolds stress model predictions can be degraded to the point where they are no better than those of the K-ε model – a state of affairs that has contributed to some of the misleading critical evaluations of second-order closures that have been published in the recent past.

It must be remembered that second-order closure models are *one-point closures* and thus can never yield accurate quantitative predictions for a wide variety of turbulent flows where the energy spectrum can change drastically. Nonetheless, with the implementation of the improvements discussed in this paper, it is quite possible that a new generation of second-order closures can be developed that will provide acceptable engineering answers for a significant range of turbulent flows that are of technological interest.

Acknowledgement

This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18605 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665.

References

[1] J. C. Rotta: Z. Phys. **129**, 547 (1951).

[2] B. E. Launder, D. P. Tselepidakis, and B. A. Younis: J. Fluid Mech. **183**, 63 (1987).

[3] M. M. Gibson, W. P. Jones, and B. A. Younis: Phys. Fluids **24**, 386 (1981).

[4] C. G. Speziale: Geophys. and Astrophys. Fluid Dyn. **23**, 69 (1983).

[5] C. G. Speziale: J. Fluid Mech. **178**, 459 (1987).

[6] C. G. Speziale: Theoret. Comput. Fluid Dynamics **1**, 3 (1989).

[7] B. E. Launder and Y. M. Ying: J. Fluid Mech. **54**, 289 (1972).

[8] W. Rodi: ZAMM **56**, T219 (1976).

[9] K. S. Choi and J. L. Lumley: in Turbulence and Chaotic Phenomena in Fluids (edited by T. Tatsumi), p. 267, North Holland (1984).

[10] K. Hanjalic and B. E. Launder: J. Fluid Mech. **52**, 609 (1972).

[11] C. G. Speziale and N. Mac Giolla Mhuiris: J. Fluid Mech. (in press).

[12] S. Tavoularis and S. Corrsin: J. Fluid Mech. **104**, 311 (1981).

[13] J. Bardina, J. H. Ferziger, and W. C. Reynolds: Stanford University Technical Report TF-19 (1983).

[14] J. P. Bertoglio: AIAA J. **20**, 1175 (1982).

[15] W. C. Reynolds: Lecture Notes for Von Karman Institute, AGARD Lecture Series No. 86, NATO (1987).

[16] B. E. Launder, G. Reece, and W. Rodi: J. Fluid Mech. **68**, 537 (1975).

[17] S. Sarkar and C. G. Speziale: ICASE Report No. 89-10, NASA Langley Research Center (1989).

[18] C. G. Speziale: Quart. Appl. Math. **45**, 721 (1987).

[19] D. C. Haworth and S. B. Pope: Phys. Fluids **29**, 387 (1986).

[20] C. G. Speziale, S. Sarkar, and T. B. Gatski: ICASE Report (in preparation).

[21] W. C. Reynolds: Private Communication (1988).

[22] K. Hanjalic and B. E. Launder: J. Fluid Mech. **74**, 593 (1976).

[23] M. Wolfshtein: Israel J. Tech. **8**, 87 (1970).

[24] C. DuP. Donaldson and G. Sandri: AGARD Report No. CP-308 (1982).

[25] J. L. Lumley, Adv. Appl. Mech. **18**, 123 (1978).

[26] T.-H. Shih and J. L. Lumley: Cornell University Technical Report FDA-85-3 (1985).

[27] S. B. Pope: AIAA J. **16**, 279 (1978).

[28] K. Hanjalic and B. E. Launder: ASME J. Fluids Engng. **102**, 34 (1980).

[29] J. Bardina, J. H. Ferziger, and R. S. Rogallo: J. Fluid Mech. **154**, 321 (1985).

Equilibrium Values	Standard K−ε Model	Launder, Reece & Rodi Model	Experiments
$(b_{11})_\infty$	0	0.193	0.201
$(b_{22})_\infty$	0	-0.096	-0.147
$(b_{12})_\infty$	-0.217	-0.185	-0.150
$(SK/\varepsilon)_\infty$	4.82	5.65	6.08

Table 1. Comparison of the predictions of the standard K−ε model and the Launder, Reece, and Rodi model with the experiments of Tavoularis and Corrsin [12] on homogeneous shear flow.

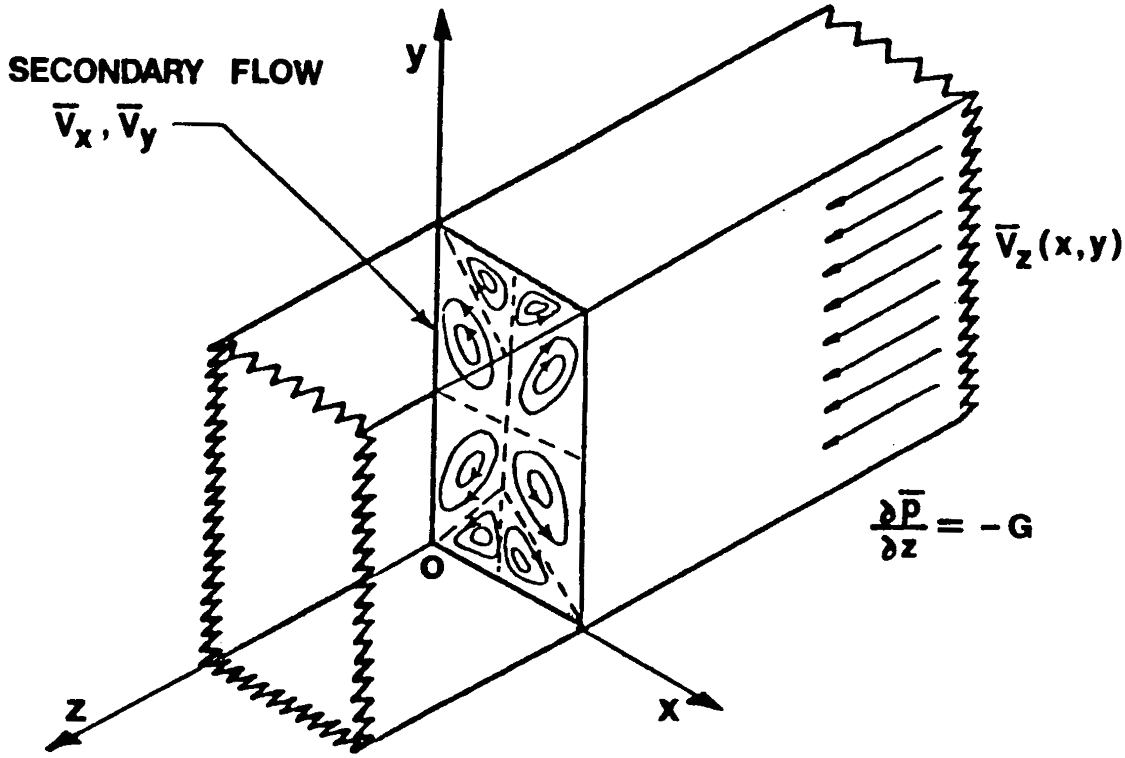

Figure 1. Turbulent secondary flow in a rectangular duct.

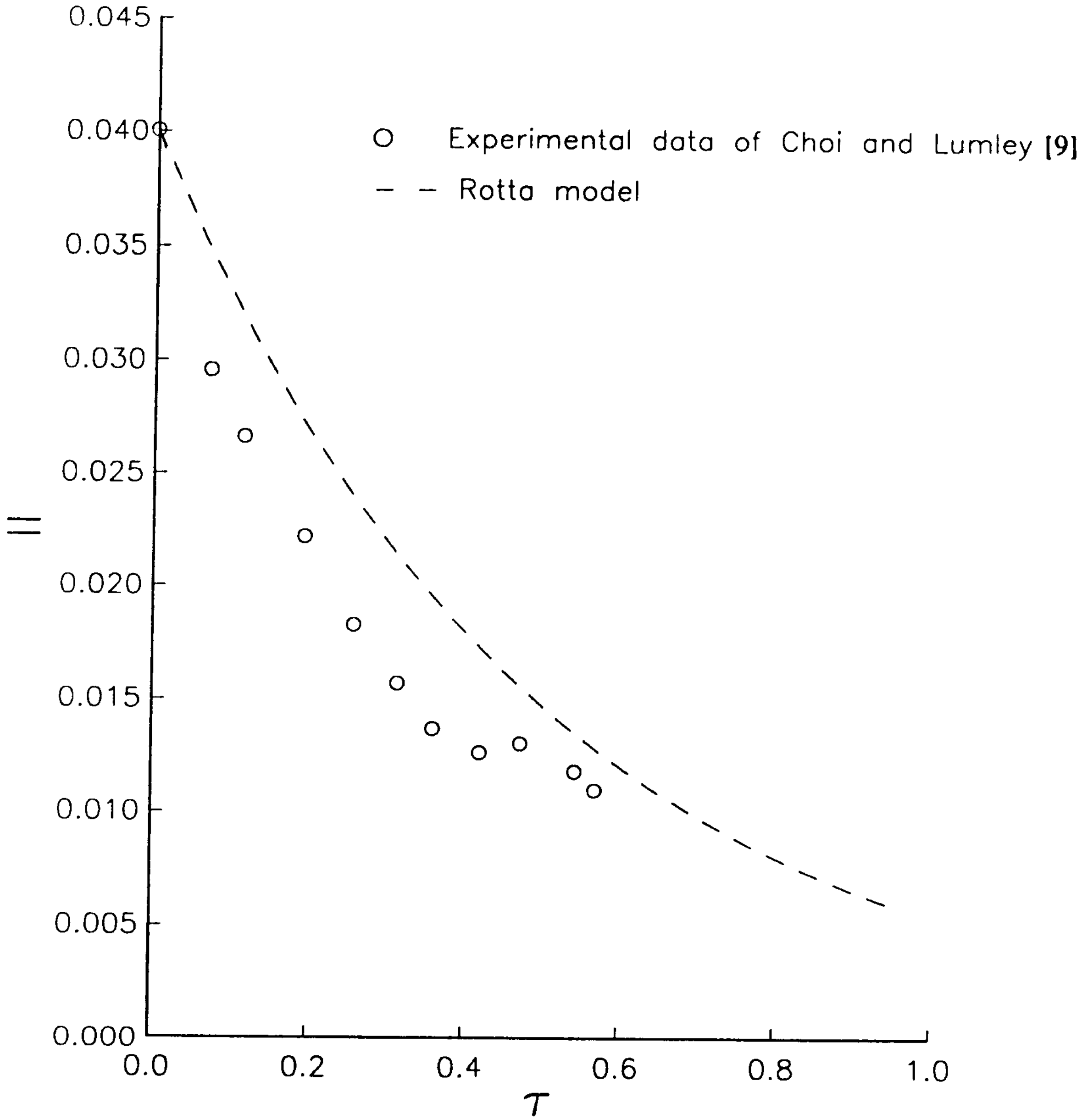

Figure 2. Temporal decay of the second invariant of the anisotropy tensor: Comparison of the Rotta model with the relaxation from plane strain experiment of Choi and Lumley [9].

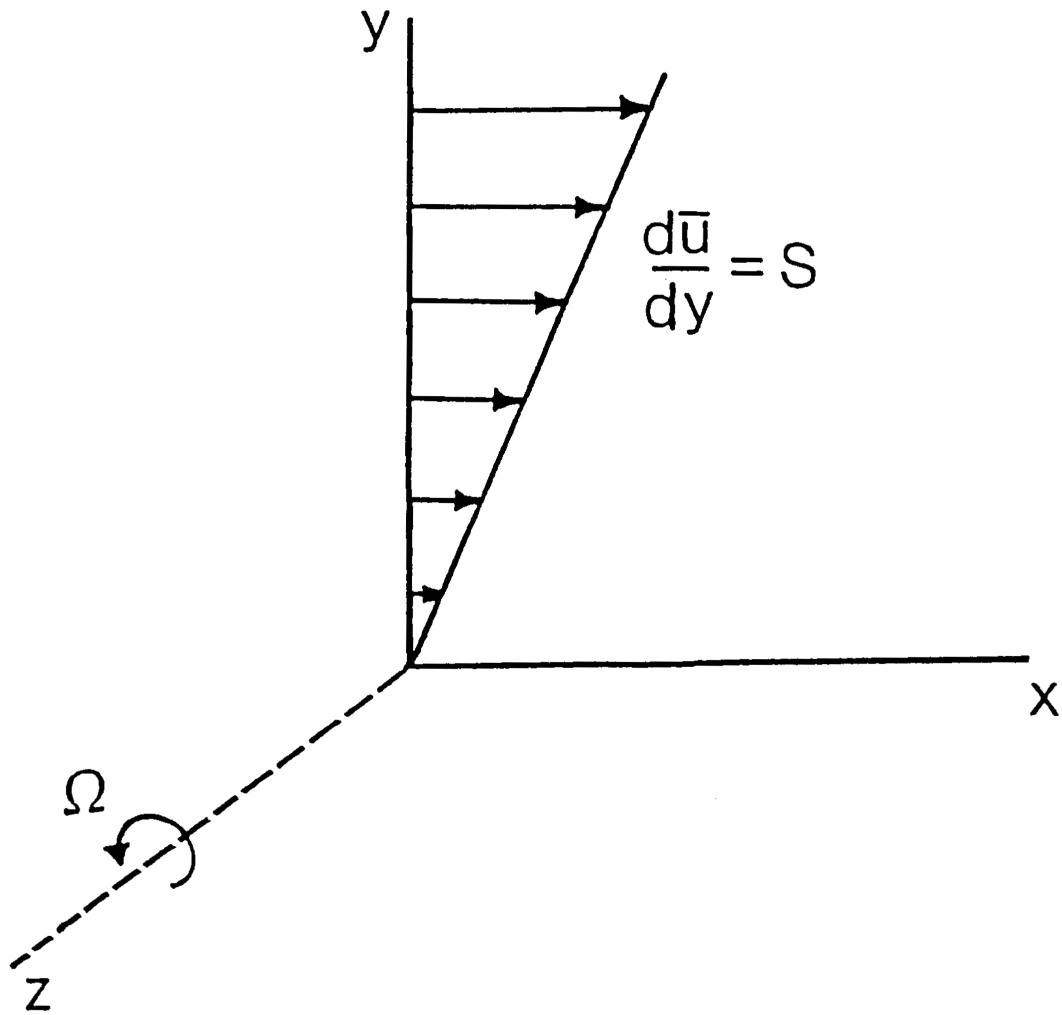

Figure 3. Homogeneous turbulent shear flow in a rotating frame.

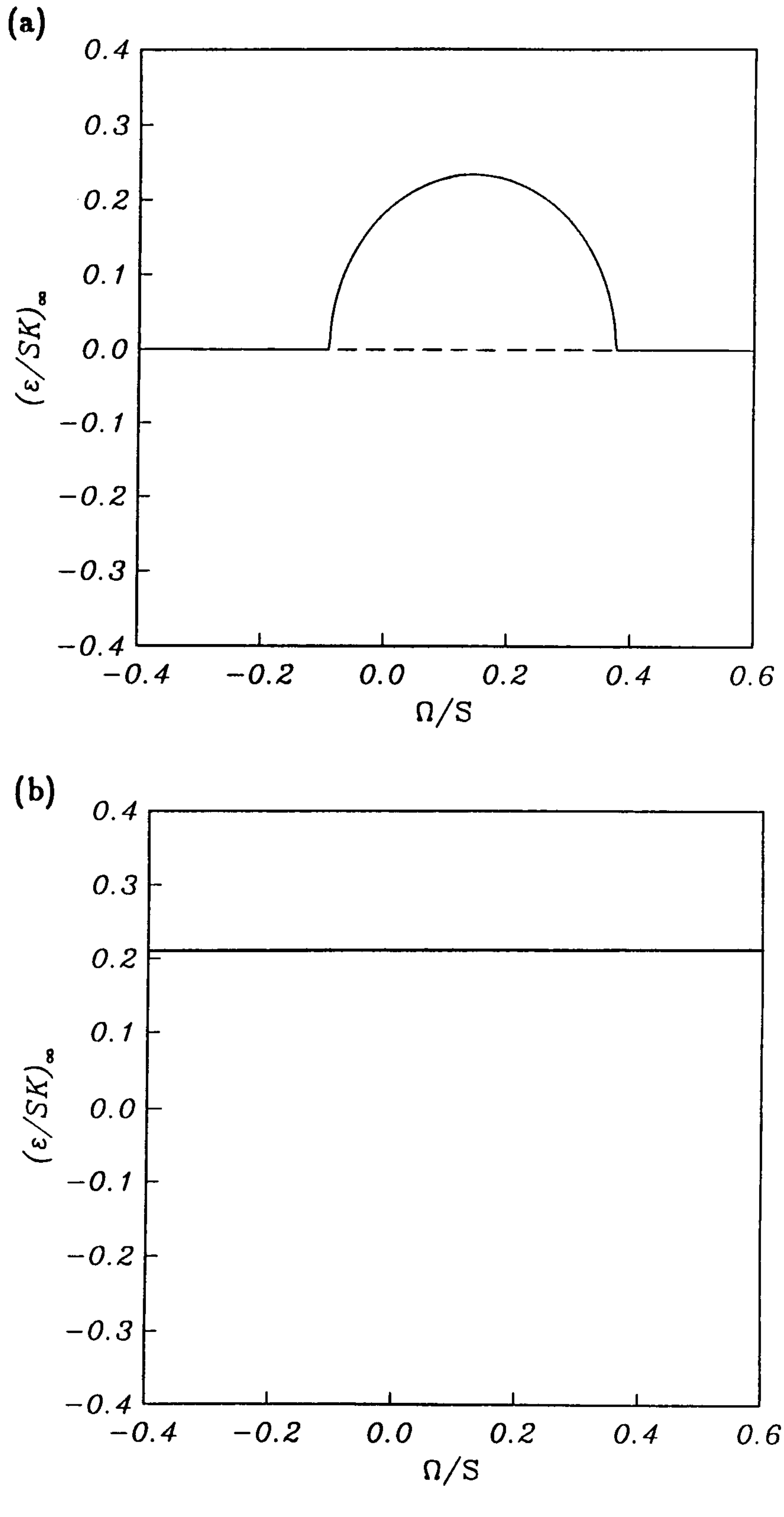

Figure 4. Bifurcation diagrams for homogeneous turbulent shear flow in a rotating frame: (a) Launder, Reece and Rodi model, (b) K-ε model.

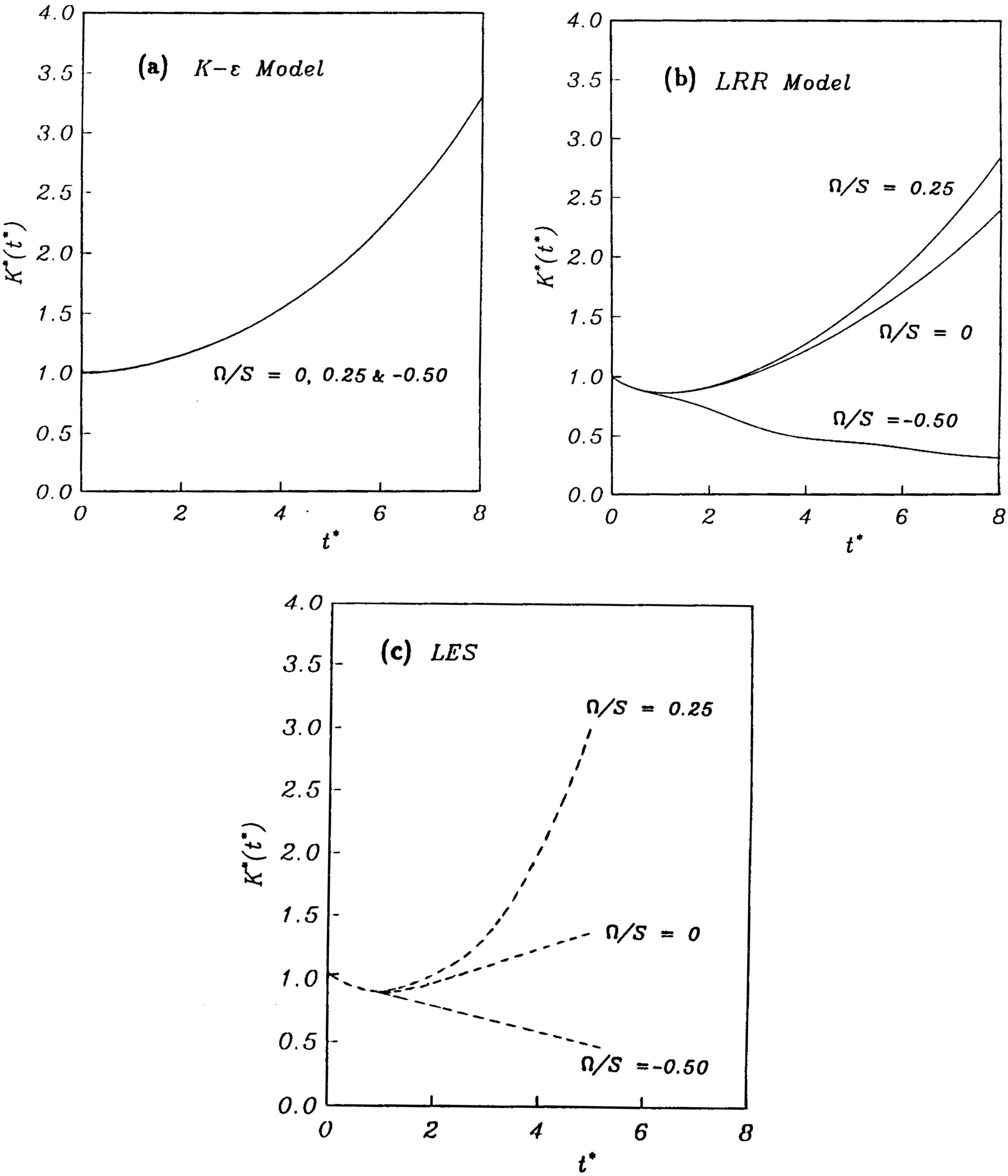

Figure 5. Time evolution of the turbulent kinetic energy for homogeneous turbulent shear flow in a rotating frame: (a) K-ε model, (b) Launder, Reece and Rodi model, (c) Large-eddy simulations of Bardina, Ferziger and Reynolds [13].

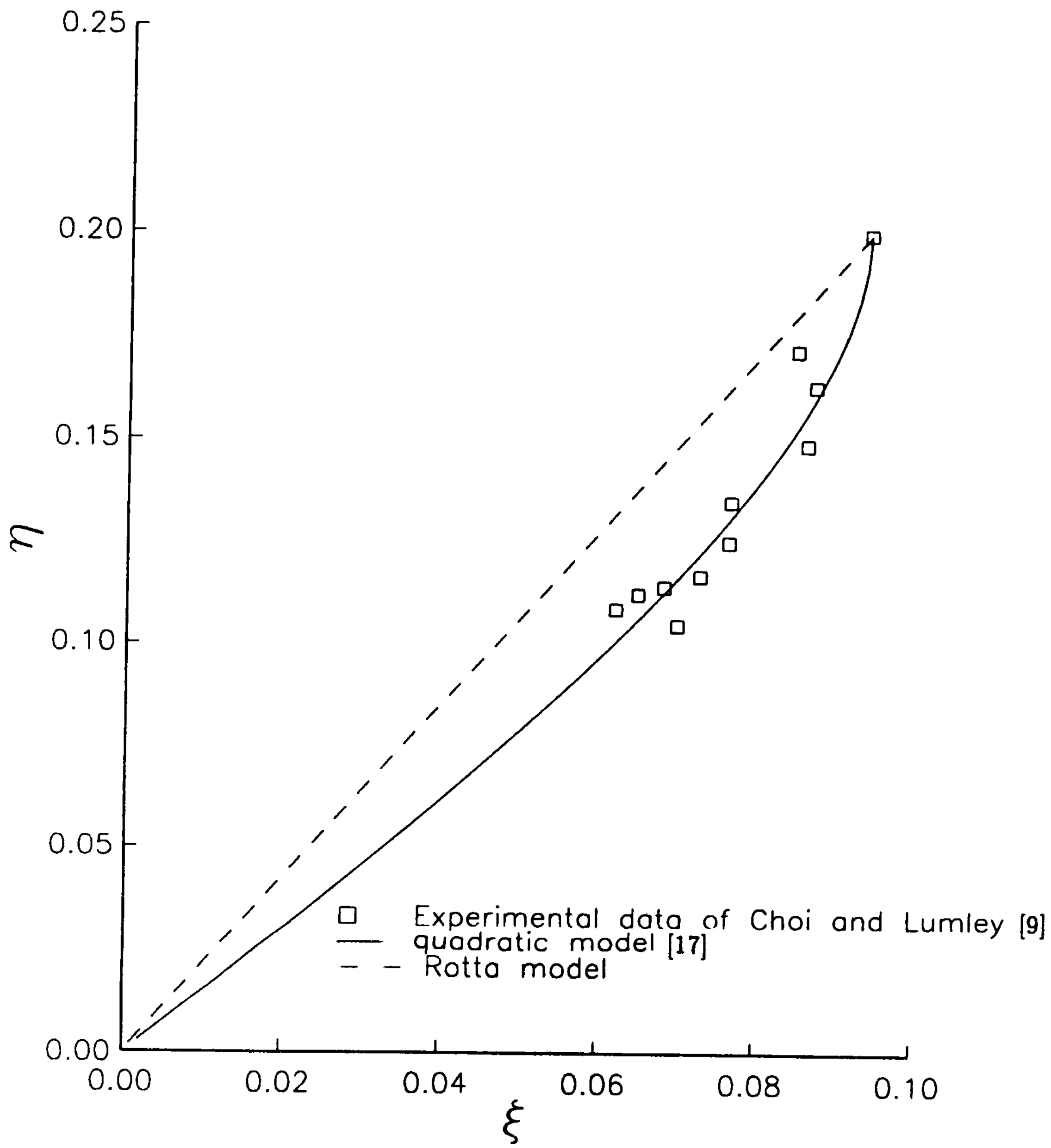

Figure 6. **Phase space diagram of the relaxation from plane strain experiment of Choi and Lumley** [9] ($\xi = III^{1/3}, \eta = II^{1/2}$).

NASA
National Aeronautics and Space Administration

Report Documentation Page

1. Report No. NASA CR-181884 ICASE Report No. 89-58	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle DISCUSSION OF TURBULENCE MODELING: PAST AND FUTURE		5. Report Date July 1989
		6. Performing Organization Code
7. Author(s) Charles G. Speziale		8. Performing Organization Report No. 89-58
		10. Work Unit No. 505-90-21-01
9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, Virginia 23665-5225		11. Contract or Grant No. NAS1-18605
		13. Type of Report and Period Covered Contractor Report
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Langley Research Center Hampton, VA 23665-5225		14. Sponsoring Agency Code

15. Supplementary Notes

Langley Technical Monitor: Richard W. Barnwell

Submitted to Lecture Notes in Physics

Final Report

16. Abstract

The full text of the discussion paper presented at the Whither Turbulence Workshop (Cornell University, March 22-24, 1989) on past and future trends in turbulence modeling is provided. It is argued that Reynolds stress models are likely to remain the preferred approach for technological applications for at least the next few decades. In general agreement with the Launder position paper, it is further argued that among the variety of Reynolds stress models in use, second-order clsoures constitute by far the most promising approach. However, some needed improvements in the specification of the turbulent length scale are emphasized. The central points of the paper are illustrated by examples from homogeneous turbulence.

17. Key Words (Suggested by Author(s)) turbulence modeling; k-ε model; second-order closure models	18. Distribution Statement 34 - Fluid Mechanics & Heat Transfer; Unclassified - Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of pages	22. Price
Unclassified	Unclassified	25	A03

NASA FORM 1626 OCT 86